An Introduction to the Lagrangian Operator

> *"When I was in high school, my physics teacher called me down one day after class and said, 'You look bored, I want to tell you something interesting'. Then he told me something I have always found fascinating. Every time the subject comes up I work on it." Richard Feynman*

Feynman's teacher told him about the "Principle of Least Action", one of the most profound results in physics and it is related to the topic that this worksheet is about to introduce. It's finally time to put the partial derivatives you just learned to use, and I can think of no better use to put them to than Lagrangian Operators. An operator, in mathematics, is an abstract mathematical object that does something. For instance, there is an operator called "del" (∇) which, if used on a function, is understood to take the partial derivative of the three directions x, y, and z, and add them all together. Our operator is actually a function all by itself, and what we do to the function is what matters. When used correctly, the Lagrangian Operator will tell us the object's equation of motion (how it will move), regardless of the difficulty of the problem.

The Lagrangian Operator
The operator itself is denoted by a cursive capital **L**, or $\mathcal{L}$.

The function that the operator contains is something oddly familiar, bringing back a topic we've certainly seen before; energy! The Lagrangian Operator is simply the difference of the Kinetic Energy and the Potential Energy of the system you're working with.

L= KE – PE

Easy enough, right? However, there is something that we must become used to when dealing with these problems, which is that the energies listed in the Lagrangian equation are the kinds of energies it will have at the moment it is moving. What this means is that, even if the problem states that the object in question isn't moving, we have to imagine the kinds of energies that it will have when it does. We need a complete picture of how the system is going to move, so we need to imagine the moment that all the energies the object can have are in play.

In reality, there are only three kinds of energies that we'll ever have to worry about:

Kinetic Energy: $\frac{1}{2}mv^2$
Gravitational Potential Energy: mgh
Elastic Potential Energy: $\frac{1}{2}kx^2$

With that in mind, making the Operator itself shouldn't be a problem. Once we've constructed the Operator, we need to take the derivative of it.

Taking Derivatives of the Operator
Remember awhile back when I mentioned that absolutely any mathematical formula could be called a function, and that we could take its derivative? Well the Lagrangian Operator is most certainly a function, and we're about to take its derivative. A lot.

There are a number of derivatives that we'll take of the Lagrangian Operator, but first we have to establish the variables we're taking the derivatives with respect to.

Generalized Position Coordinate
When we construct our Operator, KE – PE, we're always going to have at least one position (displacement, distance) coordinate, which will always be in the Potential Energy term. Whether that Potential Energy is Gravitational or Elastic, both of those energies have some position in it [Gravitational PE has h, and Elastic PE has x, both of which are positions in space]. Luckily the problems we will be dealing with will only have one of those at a time, so you won't be seeing problems with both spring and gravitational potential energy.

Because we're going to take the derivative of the Operator with respect to some position coordinate, but which position variable we use may change from problem to problem, we have something called a Generalized Position Coordinate, denoted by **q**.

One of the derivatives we'll be taking is the derivative of the operator with respect to q, and it's your job to replace q with whatever position variable you're using in this problem, whether it be x or h.

So when you see $\left(\frac{\partial L}{\partial q}\right)$ that means take the derivative of the Lagrangian Operator with respect to the variable q, where q is whatever position coordinate you're using in this problem.

Example 1: A free-falling ball of mass m will have kinetic energy and gravitational potential energy of

L = KE – PE = ½ mv^2 - mgh

So $\left(\frac{\partial L}{\partial q}\right)$ is really $\left(\frac{\partial L}{\partial h}\right)$ because height is the position that is changing

Then $\left(\frac{\partial L}{\partial h}\right)$ = 0 – mg (since h is nowhere in the first term, that whole term is a constant, and its derivative is 0, and in the second term, I use the normal way of taking a derivative, keeping m and g as constants, and get –mg as my answer).

Finally, $\left(\frac{\partial L}{\partial h}\right)$ = -mg. So we got a somewhat familiar answer from this term, the weight of the object. This will be important later, but this is how we go about taking the derivative of the Operator with respect to a "generalized position coordinate".

Generalized Velocity
In the same way that there is a generalized position, there is also a generalized velocity, but for our purposes it's much less complicated. We'll only be encountering problems with one kind of velocity (with the small exception of a particularly difficult example I'll show you), so this will be quick.

The Generalized Velocity is denoted by $\dot{q}$, or q dot. In physics, a dot above a variable indicates the rate of change of that variable with respect to time; in other words, its velocity. Since we know that q is some sort of position, we know that the rate of change of position is its velocity, so $\dot{q}$ is going to mean a velocity. When we take the derivative of the Operator with respect to $\dot{q}$, we're saying that the variable to pay attention to while taking the derivative is the velocity.

Time Derivatives
This part is simple; we just need to set a couple ground rules:

1. Mass is never changing, so we'll never be taking a derivative of mass with respect to time.
2. Remember that position, velocity, and acceleration are related by rates of change over time.

$v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$

So with all of that being said, let's get into how we put this all together.

Lagrangian Formalism
The Lagrangian Formalism is how we put all the derivatives together in a special way, such that given any problem, we can figure out how the object will move, otherwise known as its "equation of motion". At the very least we'll be able to recover some familiar laws. This equation is what we're doing all of this for, so here it is:

Deriving the Equation of Motion

$$\boxed{\left(\frac{\partial L}{\partial q}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}$$

It doesn't look like much, but it is a powerful equation.
The left side of the equation is just the partial derivative of the Lagrangian Operator with respect to the position variable, no questions there. The right side is more interesting.
On the right side we have a partial derivative of the Lagrangian Operator with respect to the velocity variable, which was introduced above. Following that on the outside of the parentheses is a lone time derivative. Here's the order we'll follow for the **right side** of the equation:

1. Take the derivative of the operator with respect to the velocity coordinate
2. Take the derivative of the RESULT of that derivative with respect to time.*

**Note: You'll notice that none of the energies we've dealt with have time in them, so what are we to do? The answer is simple: The energies we've dealt with have variables that depend on time, and their time derivative is something special. In most cases you'll take the time derivative of velocity, and what's that? Acceleration. Just be aware that you want to wedge that time derivative into your answer until it's working on something that actually is time dependent.*

Then of course, on the **left side** of the equation, we'll take the derivative with respect to the position variable, and hope to see something interesting (which we will!).

Now that we understand the formula, and we've seen some piece-by-piece examples, let's try some problems. Before we begin, let's make these clarifications:

- The Lagrangian Operator is the function that we create based on the problem we're working with: **L**= KE – PE.
- The Lagrangian Formalism is the series of derivatives that we take on the Lagrangian Operator with respect to certain variables. The Lagrangian Formalism has a set pattern and we follow it every time.
- In short, we have to build the Lagrangian Operator before we can use the Lagrangian Formalism. Building the Lagrangian Operator is just a matter of imagining what kinds of energy a system will have while it's moving, and plugging their formulae into the Lagrangian Operator.

Example 2: Applying the Lagrangian Operator and Lagrangian Formalism to the free-falling ball problem we examined earlier. We recall the Lagrangian Operator for this situation is:

L = KE – PE = ½ mv^2 - mgh

Starting on the right side of the equation, first find $\left(\frac{\partial L}{\partial \dot{q}}\right)$. Remember that $\dot{q}$ is the derivative of the position with respect to time, so we are really seeking $\left(\frac{\partial L}{\partial v}\right)$ for this problem. Taking the partial derivative of the Lagrangian with respect to velocity $\left(\frac{\partial}{\partial v}\right)\left(\frac{1}{2}mv^2 - mgh\right)$ yields **mv – 0** or **mv**. Next we find $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$ so we take the standard derivative of our previous answer to respect to time $\frac{d}{dt}(mv)$ that yields **ma**. Finally we solve the left side of the equation for $\left(\frac{\partial L}{\partial q}\right)$. Entering our Lagrangian, we obtain $\left(\frac{\partial}{\partial h}\right)\left(\frac{1}{2}mv^2 - mgh\right)$ = **-mg**, the same answer we had in Example 1. Finally, let's equate the two sides of the equation $\left(\frac{\partial L}{\partial q}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$, which gives us **-mg** = **ma** or **a** = **-g**. Not a terribly exciting result, but it is an indicator that the Lagrangian provides results that are physically significant. The falling ball does indeed accelerate at **-g**.