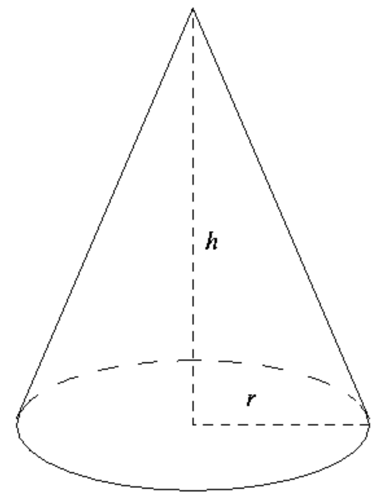
An Introduction to Partial Derivatives



Another Change in Notation

Many quantities that are interesting in mathematics and Physics, such as the volume of a cone, rely on more than one variable. We've seen in the previous worksheet how useful derivatives can be, but how can you use a derivative when there are several variables? The *partial derivative* offers an elegant solution! In a *partial derivative* we look at the change in just one variable while assuming that all other variables remain constant.

The notation for a *partial derivative* is slightly different than an ordinary derivative. Remember that in an ordinary derivative we used the notation d/dx to indicate that we were taking the derivative of the function with respect to the single variable x. When taking a partial derivative of a function with multiple variables, we use the notation δ/δx to indicate that we will be taking the partial derivative with respect to the variable **x**, assuming that all other variables remain constant. The new notation does not affect how we take the derivative; it merely indicates that there are multiple variables involved in the function.

From your previous mathematics courses you may recall that the volume of a cone can be expressed as follows:

$$V = \frac{\pi r^2 h}{3}$$

The volume of a cone depends on both the radius (***r***) and the height (***h***) so more appropriately the *function* for the volume of a cone should be written as:

$$V(r,h) = \frac{\pi r^2 h}{3}$$

We can now ask interesting questions such as how the volume of a cone relates to it height while keeping the radius constant. We will be taking the partial derivative of the volume function (V) with respect to the height variable (h), so the notation δV/δh to indicate that we are interested in the change of the cone's volume as we vary the height. We will assume that the radius (**r**) is a constant.

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\left(\frac{\pi r^2 h}{3}\right) = \frac{\partial(\pi r^2 h/3)}{\partial h} = \frac{\pi r^2}{3}$$

We can also ask how the volume varies with the radius. Again we will take a partial derivative, but this time with respect to radius (r), assuming that all other variables remain constant. We use the notation δV/δr to indicate that we are interested in the change of the cone's volume as we vary the radius. We will assume that the height (**h**) is constant.

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\pi r^2 h}{3}\right) = \frac{\partial(\pi r^2 h/3)}{\partial r} = \frac{2\pi r h}{3}$$

Another example. You may be familiar with the Ideal Gas Law that states:

PV=nRT

Rearranging this to solve for Volume we have:

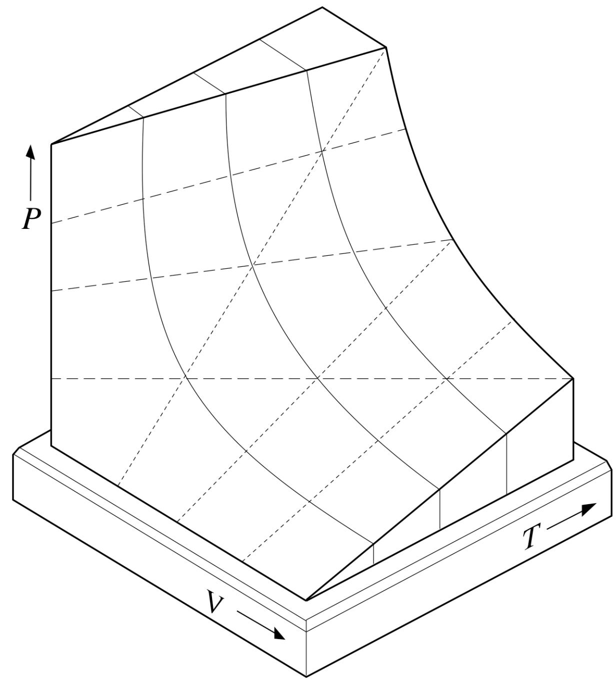
$$V = \frac{nRT}{P}$$

Or as a function of Volume:

$$V(n,T,P) = \frac{nRT}{P}$$

If we take the partial derivative of this function with respect to Temperature (T) we would get the following result:

$$\frac{\partial V}{\partial T} = \frac{\partial}{\partial T}\left(\frac{nRT}{P}\right) = \frac{nR}{P}$$



This result describes how the volume of a sample of gas varies as the Temperature is changed, but all other variables remain constant. Notice that this change δV/δT, which is the slope at a position, is a constant value. If you examine the graph shown above you will notice that Volume does indeed increase linearly with Temperature for a constant Pressure (the horizontal dashed lines) and the slope of each of these dashed lines is nR/P.

We can also examine how the Volume varies with Pressure. This is a slightly more difficult partial derivative but easily obtained:

$$\frac{\partial V}{\partial P} = \frac{\partial}{\partial P}\left(\frac{nRT}{P}\right) = \frac{\partial}{\partial P}\left(nRTP^{-1}\right) = -\frac{nRT}{P^2}$$

Notice that this change δV/δP (the parabolic curves shown) have a slope of $-nRT/P^2$ at each point.

Here are a few more general partial derivatives for you to examine.

Given a function **f** that varies with both **x** and **y**:

$$f(x,y) = xy^2 + 2x$$

Find its partial derivatives in both **x** and **y**:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2 + 2x) = y^2 + 2$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^2 + 2x) = 2xy$$

Homework Problems:

$\frac{\partial f}{\partial x}$ for the function $f(x,y) = x^3y^2 + 2xy$	$\frac{\partial f}{\partial m}$ for the function $f(m,h,v) = mgh + \frac{1}{2}mv^2$
$\frac{\partial f}{\partial y}$ for the function $f(x,y) = x^3y^2 + 2xy$	$\frac{\partial f}{\partial v}$ for the function $f(m,h,v) = \frac{1}{2}mv^2 - mgh$
$\frac{\partial f}{\partial y}$ for the function $f(x,y,z) = 4x^3y^2z$	$\frac{\partial f}{\partial h}$ for the function $f(m,h,v) = \frac{1}{2}mv^2 - mgh$
$\frac{\partial f}{\partial m}$ for the function $f(m,h) = mgh$	$\frac{\partial f}{\partial x}$ for the function $f(m,v,k,x) = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$
$\frac{\partial f}{\partial x}$ for the function $f(k,x) = \frac{1}{2}kx^2$	$\frac{\partial f}{\partial v}$ for the function $f(m,v,k,x) = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$