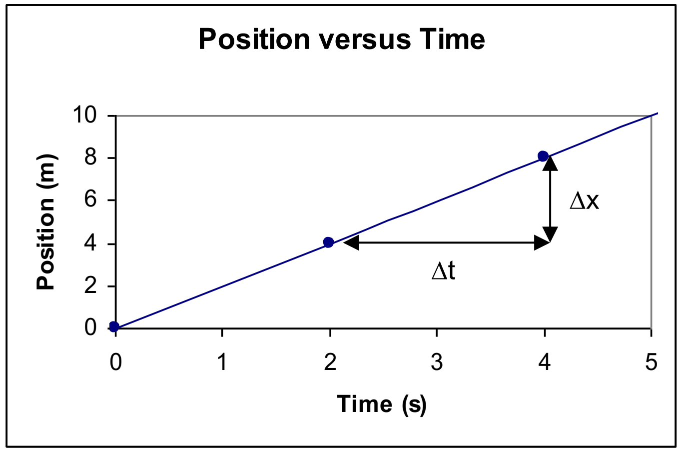
An Introduction to Derivatives



A Change in Notation:

You will remember that the slope of a ***position versus time*** graph (like the one shown to the right) will give you the velocity of an object. The slope of a graph is defined as the ***rise*** over ***run***, or the change in the ***vertical variable*** divided by the change in the ***horizontal variable***. In this particular graph, the vertical variable is the position and the horizontal variable is the time. Thus the slope of this graph can be defined as:

$$Slope = \frac{\Delta x}{\Delta t} = \frac{x_f - x_o}{t_f - t_o} = \frac{8m - 4m}{4s - 2s} = \frac{4m}{2s} = 2\,{}^{m}\!/\!{}_{s}$$

Thus the formula for the motion of this object could be described as: **x = 2t** [leaving out units (m/s) for clarity]

A ***derivative*** uses a slightly different notation, but with similar results. Instead of using the capital Greek letter delta (Δ), the derivative uses the lowercase Greek letter delta (δ) which is often written more simply as the letter *d*.

Thus the slope of this graph would be written as:

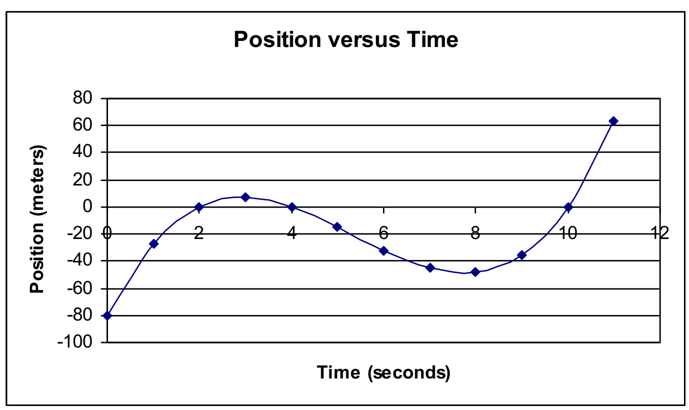
$$Slope = \frac{dx}{dt} = 2\,{}^{m}\!/\!{}_{s}$$

But this new form of the slope equation does not express the slope ***between*** two points; it actually expresses the slope at a ***single*** point. It does this by letting the change in time (Δt) approach zero – thus the change in notation to the lowercase delta. You could also find the slope of the curve at a single point manually by drawing a tangent to the curve, but as you see, derivatives offer a more elegant solution to this problem

Mathematically, the derivative of the graph above can be expressed as follows:

$$\frac{dx}{dt} = \frac{d(2t)}{dt} = 2$$

Which can expressed as the *change in the function* **2t** with respect to the *change in time* is 2. (Again leaving out the unit, m/s, for clarity)



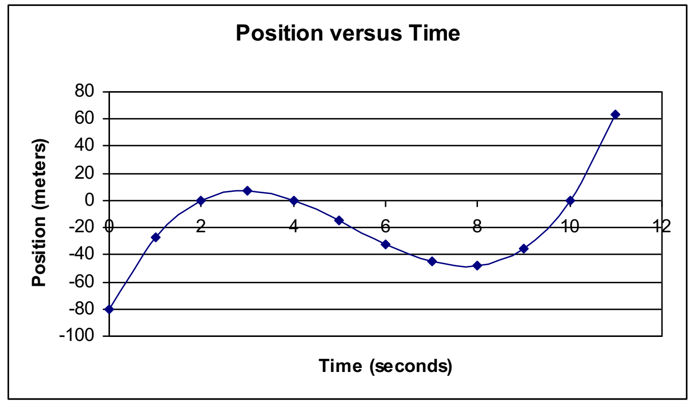
For the graph shown above, this change is not too interesting because the slope is the same (2 m/s) at ***every*** point, but it is especially useful in cases where the slope is ***not constant*** as in this next case:

Here the motion of the object is more complex, stopping and reversing direction for a portion of its journey. You can see that it stops briefly near 3 seconds and 8 seconds, but it would be nice to find the velocity at each point without resorting to drawing tangent lines at each point on the graph to find the velocity at that instant.

Derivatives offer a solution to this problem as long as the function that describes the curve is ***known***.

The function that describes the position of this object is:

$x = t^3 - 16t^2 + 68t - 80$



To find the velocity (slope) of the graph at each point, we need to take the derivative of this function. The notation for this appears as follows:

$$\frac{dx}{dt} = \frac{d(t^3 \; - \; 16t^2 \; + \; 68t \; - \; 80)}{dt}$$

This notation means that we are looking for the change in the function **$t^3 - 16t^2 + 68t - 80$** with respect to the change in time *dt*. The result of taking this derivative is:

$$\frac{d(t^3 \; - \; 16t^2 \; + \; 68t \; - \; 80)}{dt} = 3t^2 - 32t + 68$$

This means that the slope (or velocity) of this graph can be found by plugging the time into the differentiated function **$3t^2 - 32t + 68$**. For example, we can find the slope at t = 8 seconds by substituting it into our new function:

Slope $= \mathbf{3}(8)^\mathbf{2} \mathbf{- 32}(8) \mathbf{+ 68}$
$= 192 - 256 + 68$
$= 4$

So at the instant that t = 8 seconds, the slope of the graph is 4 m/s. You could also use the same function to find times when the object was stopped (the slope = 0), or determine when the object has a positive or negative velocity. The question remains however, ***how do you find a derivative?***

Calculating a derivative:

There is a simple rule to calculate the derivative of most functions. It simply requires multiplying the exponent of each term by the original coefficient and then reducing the exponent of each term by one. For example:

Taking the derivative of **t^3** with respect to the variable **t** becomes:

$$\frac{d(t^3)}{dt} = (1 \cdot 3)t^{3-1} = 3t^2$$

Or taking the derivative of **$-16t^2$** with respect to the variable **t** becomes:

$$\frac{d(16t^2)}{dt} = (-16 \cdot 2)t^{2-1} = -32t^1 = -32t$$

Or taking the derivative of **$68t$** with respect to the variable **t** becomes:

$$\frac{d(68t)}{dt} = (68 \cdot 1)t^{1-1} = 68t^0 = 68$$

Finally, the ***derivative of a constant is always defined as zero***. This may seem strange at first, but remember that the derivative tells us the slope of a function; but when a function is constant (-80), there is *no* slope. So taking the derivative of **-80** with respect to the variable **t** becomes:

$$\frac{d(-80)}{dt} = 0$$

So in our example, we found the derivative of our complex function (**t^3 – 16t^2 + 68t - 80**), by taking the derivative of each of its components:

$$\frac{d(t^3 \ - \ 16t^2 \ + \ 68t \ - \ 80)}{dt} = \frac{d(t^3)}{dt} + \frac{d(-16t^2)}{dt} + \frac{d(68t)}{dt} + \frac{d(-80)}{dt} = 3t^2 - 32t + 68 - 0$$

This process will work with most functions. Try it with the following equations:

$\frac{d(2t^2)}{dt} =$	$\frac{d(2\pi t^2)}{dt} =$
$\frac{d(2t^3)}{dt} =$	$\frac{d(4t^{-1})}{dt} =$
$\frac{d(3t^5 + 3t^2)}{dt} =$	$\frac{d(2xt^2)}{dt} =$

A note about the notation dt: The last problem you did included the variable **x**, but we are looking for changes in the function as **t** changes. Unless we know of another mathematical equation that describes how **x** changes as **t** changes, we assume that **x** is a ***constant*** in the equation just as **π** was a constant in the fourth equation listed, and treat it as simply part of the coefficient.

Multiple Order Derivatives:

Sometimes it is useful to take a **derivative of a derivative**. We know from the opening example on this worksheet that taking the derivative of **x** with respect to **t** (dx/dt) gives us the slope of the *position versus time* graph, which in Physics is defined as the velocity. You might remember that the slope of the *velocity versus time* graph is the acceleration; so if we take the derivative of **v** with respect to **t** (dv/dt), we can determine an object's acceleration at any time. Using our opening example:

$$x = \mathrm{t}^3 \text{ -16t}^2 + 68\text{t - 80}$$

$$v = \frac{dx}{dt} = \frac{d(t^3 \ - \ 16t^2 \ + \ 68t \ - \ 80)}{dt} = 3t^2 - 32t + 68 - 0$$

$$a = \frac{dv}{dt} = \frac{d(3t^2 - 32t + 68)}{dt} = 6t - 32 + 0$$

The last derivative can also be written as the **second derivative** of **x** with respect to **t**. This notation appears as follows:

$$a = \frac{d^2x}{dt^2} = \frac{d^2(t^3 \ - \ 16t^2 \ + \ 68t \ - \ 80)}{dt^2} = 6t - 32 + 0$$

This notation means that you wish to take the derivative **twice** with respect to **t**. Notice that the result is the same as before. For the function **x = 2t^4 + 4t^3 + 12t – 50,** determine the following:

v = **a =**

What is the object's position, velocity and acceleration at t = 3 seconds? x = v = a =
Is the object's acceleration increasing or decreasing at t = 3 seconds?

Special Derivatives

There are four common derivatives used in Physics that do not fit the rule given above. They are the derivatives of sine, cosine and the natural logarithms. You will need to memorize these until your Calculus class explains in depth how their derivatives are taken. Their derivatives appear below:

$$\frac{d[\sin(t)]}{dt} = \cos(t) \qquad \frac{d[\cos(t)]}{dt} = -\sin(t) \qquad \frac{d[\ln(t)]}{dt} = \frac{1}{t} \qquad \frac{d[e^t]}{dt} = e^t$$

The Product Rule:

If a function consists of two functions that are multiplied by each other, there is a simple rule to follow to find the resultant derivative:

$$\frac{d(function1 \cdot function2)}{dt} = function1 \cdot \frac{d(function2)}{dt} + function2 \cdot \frac{d(function1)}{dt}$$

For example, consider the following complex function 5sin(t) that is the product of 5 and sin(t). It can be solved using the product rule as shown:

$$\frac{d[5\sin(t)]}{dt} = \frac{d[5 \cdot \sin(t)]}{dt} = 5 \cdot \frac{d[\sin(t)]}{dt} + \cos(t) \cdot \frac{d[5]}{dt} = 5 \cdot \cos(t) + \cos(t) \cdot 0 = 5\cos(t)$$

$$\frac{d[5t \cdot \sin(t)]}{dt} = \qquad \frac{d[2\pi f \cdot \sin(t)]}{dt} =$$

$$\frac{d[\sin^2(t)]}{dt} = \frac{d[\sin(t) \cdot \sin(t)]}{dt} = \qquad \frac{d(2t \cdot \ln(t))}{dt} =$$

The Quotient Rule:

Although it is not used commonly in Physics, there is a similar rule to use when you have a quotient of two functions:

$$\frac{d\left(\frac{function1}{function2}\right)}{dt} = \frac{function2 \cdot \frac{d(function1)}{dt} - function1 \cdot \frac{d(function2)}{dt}}{(function2)^2}$$

For example, consider the derivative of the function (t^2 - 1) / (t^2 + 1). It can be solved using the quotient rule as follows:

$$\frac{d\left(\frac{t^2-1}{t^2+1}\right)}{dt} = \frac{(t^2+1) \cdot 2t - (t^2-1) \cdot 2t}{(t^2+1)^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

$$\frac{d[(t^2-1)/(t+1)]}{dt} = \qquad \frac{d[2\pi ft / \sin(t)]}{dt} =$$

$$\frac{d[\tan(t)]}{dt} = \frac{d\left[\sin(t)/\cos(t)\right]}{dt} = \qquad \frac{d(2t / \ln(t))}{dt} =$$

The Chain Rule:

You know how to take the derivative of $3t^2$ and sin(t), but how do you take the derivative of a composite function like $\sin(3t^2)$? The answer is the chain rule, probably the most commonly utilized differentiation rule used in Calculus and Physics.

The key part of this rule is to break the composite function back into two separate functions and then differentiate them together. The first step of this process is to choose the inner function and to define it temporary as **u** (u is the letter used most commonly in Calculus for this task). So for our example function, $\sin(3t^2)$, we would define the inner function, $3t^2$, as ***u***. Now the original composite function becomes **sin(u),** where $\mathbf{u=3t^2}$. Now we can take the derivative of the new function **sin(u)** with respect to **u** and the inner function, $\mathbf{3t^2}$, with respect to **t** and multiply the result to find our answer.

The reason that this works is algebra. Instead of taking the original composite function's derivative with respect to t, we do it in two parts:

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

Notice that the product of the two derivatives on the right is the same (after you cancel the *du* terms) as the original derivative.

Again remember that we set $\mathbf{u=3t^2}$, so the derivative of our composite function, $\sin(3t^2)$, becomes:

$$\frac{dx}{dt} = \frac{d[\sin(u)]}{du} \cdot \frac{d(3t^2)}{dt}$$

The derivatives of each part are:

$$\frac{d[\sin(u)]}{du} = \cos(u)$$

$$\frac{d(3t^2)}{dt} = 6t$$

So the result is **6t•cos(u)**, or $\mathbf{6t \bullet \cos(3t^2)}$ after placing our ***u*** back into the equation.

Here is another example: what is the derivative of $\sin^4(t)$? This may not appear as a composite function, but we can set **u=sin(t)** so that the original function becomes u^4. The result is as follows:

$$\frac{dx}{dt} = \frac{d(u^4)}{du} \cdot \frac{d[\sin(t)]}{dt} = 4u^3 \cdot \cos(t) = 4[\sin(t)]^3 \cdot \cos(t) = 4\sin^3(t)\cos(t)$$

$\frac{d(2t+1)^5}{dt} =$	$\frac{d[\sin^3(t)]}{dt} =$
$\frac{d(e^{2t})}{dt} =$	$\frac{d(4e^{-2t})}{dt} =$
$\frac{d[\sin(2\pi ft)]}{dt} =$	$\frac{d[2\pi f \sin(2\pi ft)]}{dt} =$